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Periodic solitary waves for two coupled nonlinear Klein–Gordon and Schrödinger equations

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Abstract

Six new periodic analytic solitary-wave solutions of two coupled nonlinear Klein–Gordon and Schrödinger equations are presented. Besides their application to optical solitary wavepairs due to cross-phase modulation or modal birefringence, their application to the (presently) unphysical regime and the significance of symmetry are discussed. For two related special sets of the two coupled equations, additional possible analytic solutions are presented.

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1. Introduction

The advantage of using solitary-wave-like input pulses for long-distance communication systems has been known for many years [1–3]. The interplay between dispersion and nonlinearity gives rise to bright solitary waves that can propagate unattenuated in the anomalous group velocity dispersion (GVD) regime, while the normal GVD regime allows an undistorted propagation of a dark solitary wave. Important new and interesting features to nonlinear pulse propagation in a fibre are introduced when coupling between the two orthogonally polarized components of an optical field or when two pulses with different wavelengths copropagate in a fibre, for which a set of two coupled nonlinear Schrödinger (CNLS) equations are used to describe their evolution. Optical solitary waves induced by cross-phase modulation, and related work on the bright–bright [4, 5], bright–dark [6–8], bright–grey [9, 10] and grey–grey [11, 12] solitary wavepairs, have been presented by many authors [3].

In this paper, we begin with a set of two coupled equations, which may be called the general two coupled nonlinear Klein–Gordon (CNKG) equations that include CNLS equations as a special case, with arbitrary coupling parameters that are more general than those that are applicable to problems in optical communication, and we present a set of six new possible periodic solitary-wave solutions that are more general than those previously obtained [2, 9, 13–21][1]. We also present a number of additional new possible periodic solitary-wave solutions for those special CNKG equations that possess certain 'symmetries'.

[1] C_2^2 for solution (II) in the third paper in [16] should be equal to $A_2^2 + \alpha^2 k^2$.

While the specific two CNLS equations with coupling parameters that are known to be applicable to the problems in optical communication and their corresponding non-periodic solitary wavepairs have rightfully received the due attention, the study of the more general set of two CNLS equations and their corresponding periodic solitary-wave solutions that we undertake in this paper, will be seen to give us a broader perspective and understanding on the possibility and generality of analytic solutions for these equations. They have mathematical as well as potential future physical implications. One of the mathematical implications is the role played by a certain two CNLS equations that are integrable [22–24] and the relationship of a certain set [21] of *N* CNLS equations with those that pass the Painlevé test [25].

The plan of this paper is as follows. Beginning with a set of two general CNKG equations, we present, in section 2, six new analytic periodic solitary-wave solutions and the conditions for the propagation of these wavepairs. The familiar bright–bright, grey–grey and bright–grey wavepairs are three special cases of these six wavepairs. The significance of these six wavepairs with regard to the possibility of their propagation in various combinations of anomalous and normal GVD regimes for a whole range of interaction values that are both physical and (presently) unphysical, is discussed. In sections 3 and 4, we present two special classes of these CNKG equations: one we call the symmetric case and the other we call the L-set, and we present for the symmetric case a new superposition periodic solution, and for the L-set ten additional analytic periodic solutions that are not special cases of the analytic solutions for the general two CNLS equations. Although only four of these ten additional periodic solutions for the L-set are new, the significance of these four in terms of 'completing' the analytic solutions for the set that has been shown to be integrable and to pass the Painlevé test is again noted [21]. In section 5, two other types of coupled equations that are known to be reducible to the two CNLS equations considered are briefly discussed. A summary is given in section 6.

2. Two CNKG equations, general case

Consider the following general set of CNKG equations for the two complex amplitudes or wavefunctions $\phi_1(z,t)$ and $\phi_2(z,t)$ as functions of position z and time t,

$$\begin{aligned} &\mathrm{i}\phi_{1z} + \alpha_1''\phi_{1zz} + \mathrm{i}\beta_1'\phi_{1t} + \beta_1''\phi_{1tt} + \kappa_1\phi_1 + R_1(|\phi_1|^2 + \nu|\phi_2|^2)\phi_1 = 0 \\ &\mathrm{i}\phi_{2z} + \alpha_2''\phi_{2zz} + \mathrm{i}\beta_2'\phi_{2t} + \beta_2''\phi_{2tt} + \kappa_2\phi_2 + R_2(\nu|\phi_1|^2 + |\phi_2|^2)\phi_2 = 0 \end{aligned} \qquad (1)$$

where $\kappa_m, \alpha_m'', \beta_m', \beta_m'', R_m$ and ν are real parameters characteristic of the medium and interaction, and where the subscripts in z and t denote derivatives with respect to z and t, and the subscripts 1 and 2 are for the two components. For problems in the optical communication, ϕ_1 and ϕ_2 denote the electric field envelopes, and ϕ_{1zz} and ϕ_{2zz} are usually neglected since ϕ_1 and ϕ_2 are assumed to be slowly varying functions of z [3]. In that case, we set $\alpha_1'' = \alpha_2'' = 0$, and equations (1) become two CNLS equations. For the case α_1'', α_2'', β_1'' and β_2'' not equal to 0, the first derivative terms with respect to z and t can be eliminated with the substitutions

$$\phi_j(z,t) = \psi_j(z,t)\exp\{-\mathrm{i}(z/\alpha_j'' + \beta_j' t/\beta_j'')/2\} \qquad j = 1, 2$$

which transform equations (1) into the following two coupled nonlinear Klein–Gordon-type equations for ψ_j,

$$\begin{aligned} &\alpha_1''\psi_{1zz} + \beta_1''\psi_{1tt} + \mu_1\psi_1 + R_1(|\psi_1|^2 + \nu|\psi_2|^2)\psi_1 = 0 \\ &\alpha_2''\psi_{1zz} + \beta_2''\psi_{2tt} + \mu_2\psi_2 + R_2(\nu|\psi_1|^2 + |\psi_2|^2)\psi_2 = 0 \end{aligned}$$

with $\mu_j = \kappa_j + 1/(4\alpha_j'') + \beta_j'^2\big/(4\beta_j'')$.

Table 1. The six possible wavepairs for the solutions of equation (1) given by equation (3): solution 1 belongs to group I, solutions 2–4 belong to group II, and solutions 5 and 6 belong to group III.

Solution		
1 or (I, 1)	$g_1 = g_2 = sn(\gamma\xi)$	
2 or (II, 1)	$g_1 = g_2 = cn(\gamma\xi)$	
3 or (II, 2)	$g_1 = g_2 = dn(\gamma\xi)$	
4 or (II, 3)	$g_1 = cn(\gamma\xi)$	$g_2 = dn(\gamma\xi)$
5 or (III, 1)	$g_1 = sn(\gamma\xi)$	$g_2 = cn(\gamma\xi)$
6 or (III, 2)	$g_1 = sn(\gamma\xi)$	$g_2 = dn(\gamma\xi)$

We consider equations (1) as our starting equations as we can easily set $\alpha_1'' = \alpha_2'' = 0$ whenever we need to consider two general CNLS equations.

Define a pair of waveforms by

$$f_m(\xi) = \left[1 - C_m^2 g_m^2(\xi)\right]^{1/2} \qquad m = 1, 2 \tag{2}$$

in which $g_m(\xi)$ is a Jacobian elliptic function $sn(\gamma\xi), cn(\gamma\xi)$ or $dn(\gamma\xi)$ of modulus k $(0 < k^2 \leqslant 1)$, and where γ is a scaling parameter, C_m^2 is an arbitrary real constant in the range $0 < C_m^2 \leqslant 1$, ξ is an abbreviation for $\xi \equiv t - z/v$, and v is the common velocity of the waves. We make the ansatz that equation (1) has solutions of the form given by

$$\begin{aligned} \phi_1(z,t) &= A_1 f_1(\xi) e_1 = A_1\left[1 - C_1^2 g_1^2(\xi)\right]^{1/2} e_1 \\ \phi_2(z,t) &= A_2 f_2(\xi) e_2 = A_2\left[1 - C_2^2 g_2^2(\xi)\right]^{1/2} e_2 \end{aligned} \tag{3}$$

where A_1 and A_2 are the amplitudes of the two travelling waves, e_m is an abbreviation for

$$e_m \equiv e^{i\theta_m} \equiv \exp(i[K_m z - \Omega_m t + \chi_m(\xi)]) \qquad m = 1, 2$$

and K_m and Ω_m are the wave number and frequency shift of the mth wave, and $\chi_m(\xi)$ is a real function of ξ. We shall present six possible analytic periodic solutions given in the form of equation (2) with $g_1(\xi)$ and $g_2(\xi)$ given in table 1. It will be seen from the later analysis that these six solutions can be divided into three groups: group I consists of one solution (1) or one kind of 'bright–bright' (BB) periodic wavepair, group II consists of three solutions (2, 3, 4) or three kinds of 'grey–grey' (GG) periodic wavepairs, and group III consists of two solutions (5, 6) or two kinds of 'bright–grey' (BG) periodic wavepairs. A periodic wavepair of the form given by $f_1(\xi) = g_2(\xi) = dn(\gamma\xi)$ was first used by Lisak *et al* [9] following their study of the bright–grey wavepair.

The derivation and a concise presentation of these analytic solutions are given in appendix A, in which a useful comparison with the one-component nonlinear Klein–Gordon equation is also given. Each of the six analytic solutions will be accompanied by six relations or conditions that must be satisfied for the solution to be valid.

Denote

$$a_j \equiv \alpha_j'' v^{-2} + \beta_j'' \qquad j = 1, 2. \tag{4}$$

The first two conditions that accompany each solution consist of the following two equations that are the same for all six solutions:

$$v^{-1} = -2\alpha_1'' K_1 v^{-1} + \beta_1' - 2\beta_1'' \Omega_1 = -2\alpha_2'' K_2 v^{-1} + \beta_2' - 2\beta_2'' \Omega_2.$$

We now present each of the six possible solitary-wave solutions of equation (1) arranged in the order given in table 1 in the form given by equation (3), and the expression for $\chi_j(\xi)$, followed by conditions 3–6 of the six conditions that accompany each solution. They are then followed by the expressions for the amplitudes of the wavepair obtained by solving the algebraic equations given by conditions 5 and 6.

Solution 1 (I, 1)

$$\phi_1 = A_1\big[1 - C_1^2 sn^2(\gamma\xi)\big]^{1/2}\exp(\mathrm{i}[K_1 z - \Omega_1 t + \chi_1(\xi)])$$

$$\phi_2 = A_2\big[1 - C_2^2 sn^2(\gamma\xi)\big]^{1/2}\exp(\mathrm{i}[K_2 z - \Omega_2 t + \chi_2(\xi)])$$

$$\chi_j(\xi) = \big[\big(1 - C_j^2\big)\big(1 - C_j^{-2}k^2\big)\big]^{1/2}\int^{\xi}\big[1 - C_j^2 sn^2(\gamma\xi)\big]^{-1}\,\mathrm{d}\xi$$

$$k^2 \leqslant C_j^2 \leqslant 1 \qquad j = 1, 2$$

$$a_1^{-1}\big[\kappa_1 - K_1 + \beta_1'\Omega_1 - \alpha_1'' K_1^2 - \beta_1''\Omega_1^2 + \nu R_1 A_2^2\big(1 - C_2^2/C_1^2\big)\big] = \big[1 - \big(3C_1^{-2} - 1\big)k^2\big]\gamma^2$$

$$a_2^{-1}\big[\kappa_2 - K_2 + \beta_2'\Omega_2 - \alpha_2'' K_2^2 - \beta_2''\Omega_2^2 + \nu R_2 A_1^2\big(1 - C_1^2/C_2^2\big)\big] = \big[1 - \big(3C_2^{-2} - 1\big)k^2\big]\gamma^2$$

$$a_1^{-1}R_1\big(A_1^2 + \nu A_2^2 C_2^2/C_1^2\big) = 2C_1^{-2}k^2\gamma^2$$

$$a_2^{-1}R_2\big(A_2^2 + \nu A_1^2 C_1^2/C_2^2\big) = 2C_2^{-2}k^2\gamma^2.$$

The last two equations can be solved for A_1^2 and A_2^2, giving for $\nu^2 \neq 1$,

$$A_1^2 = 2C_1^{-2}k^2\gamma^2\big(a_1 R_1^{-1} - \nu a_2 R_2^{-1}\big)/(1 - \nu^2)$$

$$A_2^2 = 2C_2^{-2}k^2\gamma^2\big(a_2 R_2^{-1} - \nu a_1 R_1^{-1}\big)/(1 - \nu^2);$$

and for $\nu^2 = 1$,

$$A_1^2 + \nu\big(C_2^2/C_1^2\big)A_2^2 = 2C_1^{-2}k^2\gamma^2 a_1 R_1^{-1} = 2\nu C_1^{-2}k^2\gamma^2 a_2 R_2^{-1}.$$

Solution 2 (II, 1)

$$\phi_1 = A_1\big[1 - C_1^2 cn^2(\gamma\xi)\big]^{1/2}\exp(\mathrm{i}[K_1 z - \Omega_1 t + \chi_1(\xi)])$$

$$\phi_2 = A_2\big[1 - C_2^2 cn^2(\gamma\xi)\big]^{1/2}\exp(\mathrm{i}[K_2 z - \Omega_2 t + \chi_2(\xi)])$$

$$\chi_j(\xi) = \big\{\big(1 - C_j^2\big)\big[1 + C_j^{-2}k^2\big(1 - C_j^2\big)\big]\big\}^{1/2}\int^{\xi}\big[1 - C_j^2 cn^2(\gamma\xi)\big]^{-1}\,\mathrm{d}\xi$$

$$0 < C_j^2 \leqslant 1 \qquad j = 1, 2$$

$$a_1^{-1}\big[\kappa_1 - K_1 + \beta_1'\Omega_1 - \alpha_1'' K_1^2 - \beta_1''\Omega_1^2 + \nu R_1 A_2^2\big(1 - C_2^2/C_1^2\big)\big] = \big[1 + \big(3C_1^{-2} - 2\big)k^2\big]\gamma^2$$

$$a_2^{-1}\big[\kappa_2 - K_2 + \beta_2'\Omega_2 - \alpha_2'' K_2^2 - \beta_2''\Omega_2^2 + \nu R_2 A_1^2\big(1 - C_1^2/C_2^2\big)\big] = \big[1 + \big(3C_2^{-2} - 2\big)k^2\big]\gamma^2$$

$$a_1^{-1}R_1\big(A_1^2 + \nu A_2^2 C_2^2/C_1^2\big) = -2C_1^{-2}k^2\gamma^2$$

$$a_2^{-1}R_2\big(A_2^2 + \nu A_1^2 C_1^2/C_2^2\big) = -2C_2^{-2}k^2\gamma^2.$$

The last two equations can be solved for A_1^2 and A_2^2, giving for $\nu^2 \neq 1$,

$$A_1^2 = -2C_1^{-2}k^2\gamma^2\big(a_1 R_1^{-1} - \nu a_2 R_2^{-1}\big)/(1 - \nu^2)$$

$$A_2^2 = -2C_2^{-2}k^2\gamma^2\big(a_2 R_2^{-1} - \nu a_1 R_1^{-1}\big)/(1 - \nu^2);$$

and for $\nu^2 = 1$,

$$A_1^2 + \nu\left(C_2^2/C_1^2\right)A_2^2 = -2C_1^{-2}k^2\gamma^2 a_1 R_1^{-1} = -2\nu C_1^{-2}k^2\gamma^2 a_2 R_2^{-1}.$$

Solution 3 (II, 2)

$$\phi_1 = A_1\left[1 - C_1^2 dn^2(\gamma\xi)\right]^{1/2}\exp(\mathrm{i}[K_1 z - \Omega_1 t + \chi_1(\xi)])$$
$$\phi_2 = A_2\left[1 - C_2^2 dn^2(\gamma\xi)\right]^{1/2}\exp(\mathrm{i}[K_2 z - \Omega_2 t + \chi_2(\xi)])$$
$$\chi_j(\xi) = \left\{\left(1 - C_j^2\right)\left[k^2 + C_j^{-2}\left(1 - C_j^2\right)\right]\right\}^{1/2}\int^{\xi}\left[1 - C_j^2 dn^2(\gamma\xi)\right]^{-1}\mathrm{d}\xi$$
$$0 < C_j^2 \leqslant 1 \qquad j = 1, 2$$

$$a_1^{-1}\left[\kappa_1 - K_1 + \beta_1'\Omega_1 - \alpha_1'' K_1^2 - \beta_1''\Omega_1^2 + \nu R_1 A_2^2\left(1 - C_2^2/C_1^2\right)\right] = \left[k^2 + \left(3C_1^{-2} - 2\right)\right]\gamma^2$$

$$a_2^{-1}\left[\kappa_2 - K_2 + \beta_2'\Omega_2 - \alpha_2'' K_2^2 - \beta_2''\Omega_2^2 + \nu R_2 A_1^2\left(1 - C_1^2/C_2^2\right)\right] = \left[k^2 + \left(3C_2^{-2} - 2\right)\right]\gamma^2$$

$$a_1^{-1}R_1\left(A_1^2 + \nu A_2^2 C_2^2/C_1^2\right) = -2C_1^{-2}\gamma^2$$

$$a_2^{-1}R_2\left(A_2^2 + \nu A_1^2 C_1^2/C_2^2\right) = -2C_2^{-2}\gamma^2.$$

The last two equations can be solved for A_1^2 and A_2^2, giving for $\nu^2 \neq 1$,

$$A_1^2 = -2C_1^{-2}\gamma^2\left(a_1 R_1^{-1} - \nu a_2 R_2^{-1}\right)/(1 - \nu^2)$$
$$A_2^2 = -2C_2^{-2}\gamma^2\left(a_2 R_2^{-1} - \nu a_1 R_1^{-1}\right)/(1 - \nu^2);$$

and for $\nu^2 = 1$,

$$A_1^2 + \nu\left(C_2^2/C_1^2\right)A_2^2 = -2C_1^{-2}\gamma^2 a_1 R_1^{-1} = -2\nu C_1^{-2}\gamma^2 a_2 R_2^{-1}.$$

Solution 4 (II, 3)

$$\phi_1 = A_1\left[1 - C_1^2 cn^2(\gamma\xi)\right]^{1/2}\exp(\mathrm{i}[K_1 z - \Omega_1 t + \chi_1(\xi)])$$
$$\phi_2 = A_2\left[1 - C_2^2 dn^2(\gamma\xi)\right]^{1/2}\exp(\mathrm{i}[K_2 z - \Omega_2 t + \chi_2(\xi)])$$
$$\chi_1(\xi) = \left\{\left(1 - C_1^2\right)\left[1 + C_1^{-2}k^2\left(1 - C_1^2\right)\right]\right\}^{1/2}\int^{\xi}\left[1 - C_1^2 cn^2(\gamma\xi)\right]^{-1}\mathrm{d}\xi$$
$$\chi_2(\xi) = \left\{\left(1 - C_2^2\right)\left[k^2 + C_2^{-2}\left(1 - C_2^2\right)\right]\right\}^{1/2}\int^{\xi}\left[1 - C_2^2 dn^2(\gamma\xi)\right]^{-1}\mathrm{d}\xi$$
$$0 < C_j^2 \leqslant 1 \qquad j = 1, 2$$

$$a_1^{-1}\left\{\kappa_1 - K_1 + \beta_1'\Omega_1 - \alpha_1'' K_1^2 - \beta_1''\Omega_1^2 + \nu R_1 A_2^2\left[1 - C_2^2 + k^2\left(C_2^2 - C_2^2/C_1^2\right)\right]\right\}$$
$$= \left[1 + \left(3C_1^{-2} - 2\right)k^2\right]\gamma^2$$

$$a_2^{-1}\left\{\kappa_2 - K_2 + \beta_2'\Omega_2 - \alpha_2'' K_2^2 - \beta_2''\Omega_2^2 - \nu R_2 k^{-2} C_1^2 C_2^{-2} A_1^2\left[1 - C_2^2 + k^2\left(C_2^2 - C_2^2/C_1^2\right)\right]\right\}$$
$$= \left[\left(3C_2^{-2} - 2\right) + k^2\right]\gamma^2$$

$$a_1^{-1}R_1\left(A_1^2 + \nu k^2 A_2^2 C_2^2/C_1^2\right) = -2C_1^{-2}k^2\gamma^2$$

$$a_2^{-1}R_2\left[A_2^2 + \nu k^{-2} A_1^2 C_1^2/C_2^2\right) = -2C_2^{-2}\gamma^2.$$

The last two equations can be solved for A_1^2 and A_2^2, giving for $\nu^2 \neq 1$,

$$A_1^2 = -2C_1^{-2}k^2\gamma^2\left(a_1R_1^{-1} - \nu a_2R_2^{-1}\right)/(1-\nu^2)$$

$$A_2^2 = -2C_2^{-2}\gamma^2\left(a_2R_2^{-1} - \nu a_1R_1^{-1}\right)/(1-\nu^2);$$

and for $\nu^2 = 1$,

$$A_1^2 + \nu k^2\left(C_2^2/C_1^2\right)A_2^2 = -2C_1^{-2}k^2\gamma^2a_1R_1^{-1} = -2\nu C_1^{-2}k^2\gamma^2a_2R_2^{-1}.$$

Solution 5 (III, 1)

$$\phi_1 = A_1\left[1 - C_1^2sn^2(\gamma\xi)\right]^{1/2}\exp(\mathrm{i}[K_1z - \Omega_1t + \chi_1(\xi)])$$

$$\phi_2 = A_2\left[1 - C_2^2cn^2(\gamma\xi)\right]^{1/2}\exp(\mathrm{i}[K_2z - \Omega_2t + \chi_2(\xi)])$$

$$\chi_1(\xi) = \left[\left(1 - C_1^2\right)\left(1 - C_1^{-2}k^2\right)\right]^{1/2}\int^{\xi}\left[1 - C_1^2sn^2(\gamma\xi)\right]^{-1}\mathrm{d}\xi$$

$$\chi_2(\xi) = \left\{\left(1 - C_2^2\right)\left[1 + C_2^{-2}k^2\left(1 - C_2^2\right)\right]\right\}^{1/2}\int^{\xi}\left[1 - C_2^2cn^2(\gamma\xi)\right]^{-1}\mathrm{d}\xi$$

$$k^2 \leqslant C_1^2 \leqslant 1 \qquad 0 < C_2^2 \leqslant 1$$

$$a_1^{-1}\left[\kappa_1 - K_1 + \beta_1'\Omega_1 - \alpha_1''K_1^2 - \beta_1''\Omega_1^2 + \nu R_1A_2^2\left(1 - C_2^2 + C_2^2/C_1^2\right)\right]$$
$$= \left[1 - \left(3C_1^{-2} - 1\right)k^2\right]\gamma^2$$

$$a_2^{-1}\left[\kappa_2 - K_2 + \beta_2'\Omega_2 - \alpha_2''K_2^2 - \beta_2''\Omega_2^2 + \nu R_2A_1^2C_1^2C_2^{-2}\left(1 - C_2^2 + C_2^2/C_1^2\right)\right]$$
$$= \left[1 + \left(3C_2^{-2} - 2\right)\right]\gamma^2$$

$$a_1^{-1}R_1\left(A_1^2 - \nu C_2^2A_2^2/C_1^2\right) = 2C_1^{-2}k^2\gamma^2$$

$$a_2^{-1}R_2\left(A_2^2 - \nu C_1^2A_1^2/C_2^2\right) = -2C_2^{-2}k^2\gamma^2.$$

The last two equations can be solved for A_1^2 and A_2^2, giving for $\nu^2 \neq 1$,

$$A_1^2 = 2C_1^{-2}k^2\gamma^2\left(a_1R_1^{-1} - \nu a_2R_2^{-1}\right)/(1-\nu^2)$$

$$A_2^2 = -2C_2^{-2}k^2\gamma^2\left(a_2R_2^{-1} - \nu a_1R_1^{-1}\right)/(1-\nu^2);$$

and for $\nu^2 = 1$,

$$A_1^2 - \nu\left(C_2^2/C_1^2\right)A_2^2 = 2C_1^{-2}k^2\gamma^2a_1R_1^{-1} = 2\nu C_1^{-2}k^2\gamma^2a_2R_2^{-1}.$$

Solution 6 (III, 2)

$$\phi_1 = A_1\left[1 - C_1^2sn^2(\gamma\xi)\right]^{1/2}\exp(\mathrm{i}[K_1z - \Omega_1t + \chi_1(\xi)])$$

$$\phi_2 = A_2\left[1 - C_2^2dn^2(\gamma\xi)\right]^{1/2}\exp(\mathrm{i}([K_2z - \Omega_2t + \chi_2(\xi)])$$

$$\chi_1(\xi) = \left[\left(1 - C_1^2\right)\left(1 - C_1^{-2}k^2\right)\right]^{1/2}\int^{\xi}\left[1 - C_1^2sn^2(\gamma\xi)\right]^{-1}\mathrm{d}\xi$$

$$\chi_2(\xi) = \left\{\left(1 - C_2^2\right)\left[k^2 + C_2^{-2}\left(1 - C_2^2\right)\right]\right\}^{1/2}\int^{\xi}\left[1 - C_2^2dn^2(\gamma\xi)\right]^{-1}\mathrm{d}\xi$$

$$k^2 \leqslant C_1^2 \leqslant 1 \qquad 0 < C_2^2 \leqslant 1$$

$$a_1^{-1}\left[\kappa_1 - K_1 + \beta_1'\Omega_1 - \alpha_1''K_1^2 - \beta_1''\Omega_1^2 + \nu R_1A_2^2\left(1 - C_2^2 + k^2C_2^2/C_1^2\right)\right]$$
$$= \left[1 - \left(3C_1^{-2} - 1\right)k^2\right]\gamma^2$$

$$a_2^{-1}\big[\kappa_2 - K_2 + \beta_2'\Omega_2 - \alpha_2'' K_2^2 - \beta_2''\Omega_2^2 + \nu R_2 k^{-2} A_1^2 C_1^2 C_2^{-2}\big(1 - C_2^2 + k^2 C_2^2/C_1^2\big)\big]$$
$$= \big[k^2 + \big(3C_2^{-2} - 2\big)\big]\gamma^2$$

$$a_1^{-1} R_1\big(A_1^2 - \nu k^2 C_2^2 A_2^2/C_1^2\big) = 2C_1^{-2} k^2\gamma^2$$

$$a_2^{-1} R_2\big(A_2^2 - \nu k^{-2} C_1^2 A_1^2/C_2^2\big) = -2C_2^{-2}\gamma^2.$$

The last two equations can be solved for A_1^2 and A_2^2, giving for $\nu^2 \neq 1$,

$$A_1^2 = 2C_1^{-2}k^2\gamma^2\big(a_1 R_1^{-1} - \nu a_2 R_2^{-1}\big)/(1-\nu^2),$$
$$A_2^2 = -2C_2^{-2}\gamma^2\big(a_2 R_2^{-1} - \nu a_1 R_1^{-1}\big)/(1-\nu^2);$$

and for $\nu^2 = 1$,

$$A_1^2 - \nu k^2\big(C_2^2/C_1^2\big)A_2^2 = 2C_1^{-2}k^2\gamma^2 a_1 R_1^{-1} = 2\nu C_1^{-2}k^2\gamma^2 a_2 R_2^{-1}.$$

The six conditions that accompany each solution may at first sight seem to be complicated because of the number. However, it is convenient to group them into three pairs and view them as follows: the first two relations express the velocity v in terms of K_j and Ω_j, $j = 1, 2$; the next two relations express K_j and Ω_j in terms of A_j^2 and C_j^2; and the last two equations express A_j^2 in terms of C_j^2 and γ^2. By considering C_j^2 and γ^2 as the independent variables, we can work backwards starting from the last two equations to determine A_j^2, K_j and Ω_j in terms of them. These relations involve the constant parameters that appear in equation (1). Together with the requirement that A_1^2 and A_2^2 must be >0, these six relations place, among other things, restrictions on the range of parameters and validity of solutions.

To complete the analytic expressions for the above six solutions, we note that $\chi_j(\xi)$ is an incomplete elliptic integral of the third kind that is expressible analytically in terms of the elliptic θ-functions [26]. In particular, $\chi_j(\xi)$ can be expressed simply for certain specific values of C_j^2, and we present them below:

(1) For $g_j(\xi) = sn(\gamma\xi, k)$ or $f_j(\xi) = \big[1 - C_j^2 sn^2(\gamma\xi, k)\big]^{1/2}$, with $C_j^2 = k$, $\chi_j(\xi) = \frac{1}{2}\big\{\big(1 - C_j^2\big)\gamma\xi + \tan^{-1}\big[\big(1 - C_j^2\big)\frac{sn(\gamma\xi,k)}{cn(\gamma\xi,k)dn(\gamma\xi,k)}\big]\big\}$.

(2) For $g_j(\xi) = cn(\gamma\xi, k)$ or $f_j(\xi) = \big[1 - C_j^2 cn^2(\gamma\xi, k)\big]^{1/2}$, with $C_j^2 = k/(1+k)$, $\chi_j(\xi) = \frac{1}{2}\big\{\frac{\gamma\xi}{1-C_j^2} + \tan^{-1}\big[\frac{1}{1-C_j^2}\frac{sn(\gamma\xi,k)}{cn(\gamma\xi,k)dn(\gamma\xi,k)}\big]\big\}$.

(3) For $g_j(\xi) = dn(\gamma\xi, k)$ or $f_j(\xi) = \big[1 - C_j^2 dn^2(\gamma\xi, k)\big]^{1/2}$, with $C_j^2 = 1/(1+k)$, $\chi_j(\xi) = \frac{1}{2}\big\{\frac{\gamma\xi}{C_j^2} + \tan^{-1}\big[\frac{1}{C_j^2}\frac{sn(\gamma\xi,k)}{cn(\gamma\xi,k)dn(\gamma\xi,k)}\big]\big\}$.

The special cases of $f_j(\xi) = sn(\gamma\xi), cn(\gamma\xi)$ or $dn(\gamma\xi)$ can be easily obtained from those of $g_j(\xi) = sn(\gamma\xi), cn(\gamma\xi)$ or $dn(\gamma\xi)$ by noting the relations $sn^2(\gamma\xi) + cn^2(\gamma\xi) = 1$ and $k^2 sn^2(\gamma\xi) + dn^2(\gamma\xi) = 1$, and by assigning special values such as 1 or k^2 to C_j^2. Note that for the case $g_1(\xi) = g_2(\xi)$, the two waveforms $f_1(\xi)$ and $f_2(\xi)$ need not be the same in general depending on the values of C_1^2 and C_2^2. For example, for $C_1^2 = 1, C_2^2 = k^2$ for $g_1(\xi) = g_2(\xi) = sn(\gamma\xi)$, we have two different waveforms $f_1(\xi) = cn(\gamma\xi)$ and $f_2(\xi) = dn(\gamma\xi)$.

By noting that for $k^2 = 1, sn(\gamma\xi, 1) = \tanh(\gamma\xi), cn(\gamma\xi, 1) = dn(\gamma\xi, 1) = \mathrm{sech}(\gamma\xi)$, the six solutions coalesce into three following solutions A, B and C, where solution 1 becomes solution A, solutions 2, 3 and 4 become solution B, and solutions 5 and 6 become solution C:

Solution A

$$\phi_1 = A_1\,\mathrm{sech}(\gamma\xi)\exp(\mathrm{i}(K_1 z - \Omega_1 t))$$
$$\phi_2 = A_2\,\mathrm{sech}(\gamma\xi)\exp(\mathrm{i}(K_2 z - \Omega_2 t)).$$

Solution B

$$\phi_1 = A_1\left[1 - C_1^2\,\mathrm{sech}^2(\gamma\xi)\right]^{1/2} \exp(\mathrm{i}[K_1 z - \Omega_1 t + \chi_1(\xi)])$$
$$\phi_2 = A_2\left[1 - C_2^2\,\mathrm{sech}^2(\gamma\xi)\right]^{1/2} \exp(\mathrm{i}[K_2 z - \Omega_2 t + \chi_2(\xi)])$$
$$\chi_j(\xi) = \left(C_j^{-2} - 1\right)^{-1/2} \int^{\xi} \left[1 - C_j^2\,\mathrm{sech}^2(\gamma\xi)\right]^{-1} \mathrm{d}\xi$$
$$= \left\{ \frac{\sqrt{1-C_j^2}}{C_j}\gamma\xi + \tan^{-1}\left[\frac{C_j}{\sqrt{1-C_j^2}}\tanh(\gamma\xi)\right]\right\}$$
$$0 < C_j^2 \leqslant 1 \qquad j = 1, 2.$$

Solution C

$$\phi_1 = A_1\,\mathrm{sech}(\gamma\xi)\exp(\mathrm{i}(K_1 z - \Omega_1 t))$$
$$\phi_2 = A_2\left[1 - C_2^2\,\mathrm{sech}^2(\gamma\xi)\right]^{1/2} \exp(\mathrm{i}[K_2 z - \Omega_2 t + \chi_2(\xi)])$$
$$\chi_2(\xi) = \left\{ \frac{\sqrt{1-C_2^2}}{C_2}\gamma\xi + \tan^{-1}\left[\frac{C_2}{\sqrt{1-C_2^2}}\tanh(\gamma\xi)\right]\right\}$$
$$0 < C_2^2 \leqslant 1.$$

These three solutions are the bright–bright, grey–grey and bright–grey non-periodic solitary wavepairs. Note that while we have the grey non-periodic waveform $\left[1 - C_j^2\,\mathrm{sech}^2(\gamma\xi)\right]^{1/2}$ with $C_j^2 \leqslant 1$, we do not have a non-periodic waveform represented by $\left[1 - C_j^2\tanh^2(\gamma\xi)\right]^{1/2}$ for $C_j^2 < 1$ because of the restriction $C_j^2 \geqslant k^2$ (see solutions 1, 5 and 6), and if we set $C_j^2 = k^2 = 1$, $\left[1 - C_j^2\tanh^2(\gamma\xi)\right]^{1/2}$ becomes the bright non-periodic solitary waveform $\mathrm{sech}(\gamma\xi)$, and $\chi_j(\xi) = 0$. The reduction to three special solutions for $k^2 = 1$ and the propagation possibilities to be discussed for the wavepairs given by the six solutions make it convenient to classify them into three groups: group I consists of one kind of bright–bright (BB) periodic wavepair given by solution 1; group II consists of three kinds of grey–grey (GG) periodic wavepair given by solutions 2, 3 and 4; and group III consists of two kinds of bright–grey (BG) wavepair given by solutions 5 and 6. It will be seen later that it is sometimes useful to add a fourth group that consists of two kinds of grey–bright (GB) wavepair even though they are simply the bright–grey wavepairs in reverse order.

The expressions for A_1^2 and A_2^2 given by solving the fifth and sixth conditions that accompany each solution and the simple consideration that A_1^2 and A_2^2 must be >0 can be used to sort out the permitted solutions for the given parameters αs, βs, κs, Rs and ν in equation (1). To be specific, let us assume that

$$\alpha_1'' = \alpha_2'' = 0 \qquad R_1 > 0 \qquad R_2 > 0.$$

We now have two CNLS equations. Remembering that $\beta'' > 0$ and $\beta'' < 0$ correspond to the anomalous and normal GVD regimes respectively, we present, in appendix B, the list of and conditions for possible propagation of the three groups of wavepairs in the four regimes characterized by (i) $\beta_1'' > 0, \beta_2'' > 0$; (ii) $\beta_1'' > 0, \beta_2'' < 0$; (iii) $\beta_1'' < 0, \beta_2'' > 0$; and (iv) $\beta_1'' < 0, \beta_2'' < 0$; for $-\infty < \nu < +\infty$. The four regimes will be denoted for convenience by (++), (+−), (−+), (−−).

If we restrict ourselves to the physically applicable values of $\nu \geqslant 0$ (or $\frac{2}{3} \leqslant \nu \leqslant 2$ to be precise), then the list in appendix B gives six types of possible propagations:

(1) BB in (++) for $\nu \geqslant 0$.
(2) GG in (−−) for $\nu \geqslant 0$.

(3) BG in (++) for $\nu > 0$.
(4) BG in (+−) for $0 \leqslant \nu < 1$.
(5) BG in (−+) for $\nu > 1$.
(6) BG in (−−) for $\nu > 0$.

It is seen that the six types of propagation consist of one type of propagation involving group I wavepair (bright–bright), one type of propagation involving group II wavepairs (grey–grey), but four types of propagation involving group III wavepairs (bright–grey), i.e while the bright–bright and grey–grey wavepairs are restricted to propagate in the anomalous–anomalous and normal–normal GVD regimes respectively, the bright–grey wavepairs are seen to be able to propagate in any one of the four possible combinations of anomalous and normal GVD regimes for the two waves of the wavepair (subject to the conditions given in appendix B). Note the different range permitted for ν for possibilities 4 and 5.

The apparent 'asymmetry' above is caused by restricting our consideration to $\nu \geqslant 0$. Expanding consideration of equation (1) to include negative as well as positive values of ν is not only of mathematical interest (such as the integrability of the equations discussed in [23] and [24]), but also serves to provide a better perspective on the number of possibilities and the totality of analytic solutions that may play a role in future physical applications. Thus if we consider the case for $\nu \leqslant 0$, then it can be seen from appendix B that we have in this case nine types of possible propagation that consist of four types of propagation involving group I wavepair, four types of propagation involving group II wavepairs and only one type of propagation involving group III wavepairs, as follows:

1–4. BB in (++), (+−), (−+), (−−), for $-1 < \nu \leqslant 0, \nu < 0, \nu < 0, \nu < -1$ respectively.
5–8. GG in (++), (+−), (−+), (−−), for $\nu < -1, \nu < 0, \nu < 0, -1 < \nu \leqslant 0$ respectively.
9. BG in (+−), for $\nu \leqslant 0$.

The 'symmetry' between $\nu \geqslant 0$ and $\nu \leqslant 0$ becomes even more apparent if we list grey–bright as well as bright–grey wavepairs in the counts as presented in table 2. It is seen that if we count the wavepairs BB, GG, BG and GB as shown in table 2, then there are ten possible propagations for $\nu \geqslant 0$ (second column) and ten for $\nu \leqslant 0$ (third column). Of these, BG and GB wavepairs account for eight possible propagations for $\nu \geqslant 0$, and BB and GG wavepairs account for eight possible propagations for $\nu \leqslant 0$. The fourth column combines columns 2 and 3, and shows what range of values of ν between $-\infty$ and $+\infty$ permits propagation of the given wavepairs in the given GVD regimes. The fifth column lists which of the two special values of $\nu = +1$ or -1 permits propagation for the given wavepairs in the given GVD regimes. The numbers of possible propagations for $\nu \geqslant 0$ and $\nu \leqslant 0$, $\nu > 1$ and $\nu < -1$, $\nu < 1$ and $\nu > -1$, as well as for the special values $\nu = +1$ and -1, are seen to reflect the symmetry.

After obtaining A_1^2 and A_2^2 in terms of C_1^2 and C_2^2 and γ^2, we can proceed to obtain K_1, K_2, Ω_1 and Ω_2 in terms of $A_1^2, A_2^2, C_1^2, C_2^2, \gamma^2$ and v, using conditions 1 to 4 that accompany each solution. We illustrate this with the solution given by Trillo *et al* [6] that corresponds to a special case of solution C. We interchange the indices 1 and 2 for solution 5 or 6 and set $k^2 = 1$ so that we consider a dark–bright wavepair propagating in the anomalous–normal regimes $\beta_1'' > 0, \beta_2'' < 0$ to coincide with that given in [3] and [6], and then we set $\alpha_1'' = \alpha_2'' = \kappa_1 = \kappa_2 = 0, \nu = 2, C_1^2 = 1$. Also we note that $\beta_1' = \beta_2' = 0$ is assumed in [6], but $\beta_1' = \beta_2' = 1/v_g$ is assumed in [3] (section 7.3.1), where v_g is the group velocity of the wavepair. Since the results given in both [3] and [6] appeared to have a small error in the expression for K_2, we present below the complete expressions for A_1^2, A_2^2, K_1, K_2 and v

Table 2. Possible propagations in the GVD regimes characterized by (β_1'', β_2'') for four groups of wavepairs BB, GG, BG, GB.

(β_1'', β_2'')	$\nu \geqslant 0$	$\nu \leqslant 0$	ν	$\nu = \pm 1$
(i) (++)				
BB	$\nu \geqslant 0$	$-1 < \nu \leqslant 0$	$\nu > -1$	$\nu = +1$
GG		$\nu < -1$	$\nu < -1$	
BG	$\nu > 0$		$\nu > 0$	$\nu = +1$
GB	$\nu > 0$		$\nu > 0$	$\nu = +1$
(ii) (+−)				
BB		$\nu < 0$	$\nu < 0$	$\nu = -1$
GG		$\nu < 0$	$\nu < 0$	$\nu = -1$
BG	$0 \leqslant \nu < 1$	$\nu \leqslant 0$	$\nu < 1$	$\nu = -1$
GB	$\nu > 1$		$\nu > 1$	
(iii) (−+)				
BB		$\nu < 0$	$\nu < 0$	$\nu = -1$
GG		$\nu < 0$	$\nu < 0$	$\nu = -1$
BG	$\nu > 1$		$\nu > 1$	
GB	$0 \leqslant \nu < 1$	$\nu \leqslant 0$	$\nu < 1$	$\nu = -1$
(iv) (−−)				
BB		$\nu < -1$	$\nu < -1$	
GG	$\nu \geqslant 0$	$-1 < \nu \leqslant 0$	$\nu > -1$	$\nu = +1$
BG	$\nu > 0$		$\nu > 0$	$\nu = +1$
GB	$\nu > 0$		$\nu > 0$	$\nu = +1$

for this special case of solution C for which $\phi_1 = A_1 \tanh(\gamma\xi)\exp(\mathrm{i}(K_1 z - \Omega_1 t))$, $\phi_2 = A_2 \operatorname{sech}(\gamma\xi)\exp(\mathrm{i}(K_2 z - \Omega_2 t))$:

$$A_1^2 = \left(2\beta_1'' R_1^{-1} - 4\beta_2'' R_2^{-1}\right)\gamma^2/3 \qquad A_2^2 = \left(4\beta_1'' R_1^{-1} - 2\beta_2'' R_2^{-1}\right)\gamma^2/3$$

$$K_1 = \beta_1'\Omega_1 - \beta_1''\Omega_1^2 + R_1 A_1^2 \qquad K_2 = \beta_2'\Omega_2 - \beta_2''\Omega_2^2 + R_2 A_2^2 - \beta_2''\gamma^2$$

$$v^{-1} = \beta_1' - 2\beta_1''\Omega_1 = \beta_2' - 2\beta_2''\Omega_2.$$

The simple relationships $2R_2A_1^2 - R_2A_2^2 = -2\beta_2''\gamma^2$ and $2R_1A_2^2 - R_1A_1^2 = 2\beta_1''\gamma^2$ can be used to give alternative expressions. As was pointed out in [3, 6], a striking feature of this solitary wavepair is that the dark solitary wave propagates in the anomalous GVD regime ($\beta_1'' > 0$) whereas the bright solitary wave propagates in the normal GVD regime ($\beta_2'' < 0$), exactly opposite of the behaviour expected in the absence of cross-phase modulation. As was noted in our discussion of possibility 5 of our more general periodic bright–grey solutions, a necessary condition for this so-called inverted bright–dark wavepair is $\nu > 1$, a condition that was not explicitly stated previously. As for the so-called normal bright–grey wavepair [8], a necessary condition is $0 \leqslant \nu < 1$. All this and other cases, including all the necessary conditions, are contained in the general results we present in appendix B.

3. Special case 1: the symmetric case

For certain special cases of equation (1) involving certain specific parameters characterized by α_j'', β_j', β_j'', R_j and ν, there are other types of analytic solutions in addition to the six possible analytic solutions given in the previous section. There are two special cases we want to discuss, the first of which, the 'symmetric case' [15][2] is discussed in this section.

[2] All expressions on the right-hand sides of equations (4*a*)–(4*c*) should be multiplied by γ^2 in this paper.

The ‘symmetric’ case for equation (1) refers to the case in which

$$\kappa_1 = \kappa_2 \quad \alpha_1'' = \alpha_2'' \quad \beta_1' = \beta_2' \quad \beta_1'' = \beta_2'' \quad R_1 = R_2 \quad \nu = 1. \tag{5}$$

In this case, the solution of equation (1) can be expressed more generally as a superposition solution given by

$$\phi_m(z,t) = \sum_{j=1}^{2} A_{mj} f_j(\xi) \exp\{\mathrm{i}[K_j z - \Omega_j t + \chi_j(\xi)]\} \qquad m = 1, 2 \tag{6}$$

where $A_{11}A_{12} + A_{21}A_{22} = 0$ without necessarily requiring any one of the As to be equal to zero. Here, the six possible analytic solutions of equation (1) for this symmetric case are given by equation (6) in which the pair of waveforms $f_j(\xi)$, $j = 1, 2$, is given by equation (2), with the pair of $g_j(\xi)$ given by those presented in table 1 that belong to the same solution. If we define

$$A_1^2 \equiv A_{11}^2 + A_{21}^2 \qquad A_2^2 \equiv A_{12}^2 + A_{22}^2$$

we find that the six conditions that accompany each solution given by equation (6) are exactly those that accompany solutions 1–6 in section 2. The six superimposed wavepairs can be divided into three groups, as in section 2, so are their possible propagations in various GVD regimes.

These superposition solutions for the special case in which $f_j(\xi)$ is $sn(\gamma\xi)$, $cn(\gamma\xi)$ or $dn(\gamma\xi)$ and $\chi_j(\xi) = 0$ were given by the author in [15]. Note that these superposition solutions, similar to those for the three- and five-level systems [27] for a somewhat related problem, involve superpositions of generally two *different* elliptic functions with *arbitrary* constants, and they should be distinguished from the ‘superposition’ solutions given by Cooper *et al* [28] for various (one-component) nonlinear equations that involve the *same* elliptic functions that are centred at equally spaced points with *no* arbitrary constants. As Khare and Sukhatme [29] showed in a subsequent paper, their ‘superposition’ is in fact a remarkable generalization of Landen’s quadratic transformation formulae for Jacobian elliptic functions.

4. Special case 2: the L-set

The second special case of equation (1) we shall discuss is referred to as the L-set [21] as its solutions can be analytically represented in terms of Lamé functions [30]. The L-set is interesting mathematically as the CNLS equations belonging to this set pass the Painlevé test [25]. For N CNLS equations, the L-set is given by the following (normalized) coupled equations:

$$\mathrm{i}\phi_{mz} \pm \beta_m \phi_{mtt} + \kappa_m \phi_m \pm \left(\sum_{j=1}^{N} \beta_m \beta_j |\phi_j|^2 \right) \phi_m = 0 \qquad m = 1, \ldots, N \tag{7}$$

where $\beta_j = +1$ or -1, $j = 1, \ldots, N$. If we make the substitution

$$\phi_m(z,t) = \psi_m(t)\, \mathrm{e}^{\mathrm{i}\omega_m z} \tag{8}$$

where ω_m are real constants and $\psi_m(t)$ are real functions of t only, then the coupled equations for $\psi_m(t)$ are

$$\psi_{mtt} + c_m \psi_m + \left(\sum_{j=1}^{N} \beta_j \psi_j^2 \right) \psi_m = 0 \qquad m = 1, \ldots, N \tag{9}$$

where

$$c_m = \pm\beta_m(\kappa_m - \omega_m). \quad (10)$$

To eliminate the permutation symmetry, we arrange equation (9) such that

$$c_1 \geqslant c_2 \geqslant \cdots \geqslant c_N \quad (11)$$

so that only one of the two choices (the upper or lower sign) in equations (7) and (10) corresponds to the equations of motions for equation (9). The travelling waves can be constructed (see appendix C) by substituting the solutions ψ_m from equation (9) into equation (8), and replacing $\phi_m(z, t)$ by

$$\phi_m(z, t - z/v) \exp\left\{\pm \mathrm{i}\beta_m^{-1}[t - z/(2v)]/(2v)\right\} \quad (12)$$

where v is the common velocity of the waves.

We consider equation (7) with the upper signs since the lower signs give no new physics, and we characterize the interaction parameters of equation (7) by the array $(\beta_1, \beta_2, \ldots, \beta_N)$, where $\beta_j = +1$ or -1 (or denoted simply by + or −), and refer to each of the 2^N arrays as an interaction type. This special set of N CNLS equations possesses analytic solutions [21] for $\phi_m(z, t)$ that can be expressed in terms of Lamé functions [30] of order $n \leqslant N$.

We use Lamé equation of order n expressed in the form

$$\mathrm{d}^2 f/\mathrm{d}\tau^2 + [h - n(n+1)k^2 sn^2(\tau, k)]f = 0. \quad (13)$$

We refer to the polynomial solutions (that are doubly periodic) of the Lamé equation as Lamé functions, and we shall number the $2n + 1$ Lamé functions of order n, $f_1^{(n)}, f_2^{(n)}, \ldots, f_{2n+1}^{(n)}$, in the order of numbering their corresponding eigenvalues $h_m^{(n)}$ arranged in descending order $h_1^{(n)} > h_2^{(n)} > \cdots > h_{2n+1}^{(n)}$. The use of Lamé function ansatz described in [18] gives the solutions of equation (9) in terms of Lamé functions in the form

$$\psi_m(t) = A_m f_p^{(n)}(\gamma t) \quad (14)$$

where A_m is the 'amplitude' of the mth component, $f_p^{(n)}$ is the pth Lamé function of order n, and γ is a scaling parameter. To obtain the solutions of equation (9) using the Lamé function ansatz, we assume equation (14) and express the square of the jth Lamé function of order n in a power series in $s \equiv sn(\tau, k)$ as

$$\left[f_j^{(n)}(\tau)\right]^2 = \sum_{i=1}^{n+1} a_{ij}^{(n)} s^{2(i-1)} \qquad j = 1, \ldots, 2n+1$$

and substitute these into equation (9). Comparing them with equation (13) gives a set of algebraic equations that need to be satisfied and give the required values for the amplitudes A_m and the required values of c_m for equation (9). An N-combination $\left(f_p^{(n)}, f_q^{(n)}, \ldots, f_s^{(n)}\right)$ that gives an analytic solution for the N components $(\psi_1, \psi_2, \ldots, \psi_N)$ will be represented simply by $(p, q, \ldots, s)_n$, where equation (11) implies $p \leqslant q \leqslant \cdots \leqslant s$. We first renumber the $2n+1$ eigenvalues of the Lamé equation $h_1^{(n)}, h_2^{(n)}, \ldots, h_{2n+1}^{(n)}$, as $h_1^{(n)}, h_2^{(n)}, h_{2'}^{(n)}, \ldots, h_{n+1}^{(n)}, h_{(n+1)'}^{(n)}$, and the corresponding Lamé functions $f_1^{(n)}, f_2^{(n)}, \ldots, f_{2n+1}^{(n)}$, as $f_1^{(n)}, f_2^{(n)}, f_{2'}^{(n)}, \ldots, f_{n+1}^{(n)}, f_{(n+1)'}^{(n)}$, i.e. we group them in pairs except the first one. This numbering system has been shown [21] to be very useful for identifying the possible combinations that are solutions of a given set of interaction parameters.

An N-combination $\left(f_p^{(n)}, f_q^{(n)}, f_{r'}^{(n)}, \ldots, f_s^{(n)}\right)$ for example, will now be represented by $(p, q, r', \ldots, s)_n$, where $p \leqslant q \leqslant r \leqslant \cdots \leqslant s$.

The $2n + 1$ Lamé functions $f_m^{(n)}(\tau)$ and their eigenvalues $h_m^{(n)}$ satisfy the Lamé equation (13), and we list them according to the subscript $m = 1, 2, 2', 3, 3', \ldots, n+1, (n+1)'$ for $n = 1$ and 2, and they are given in table 3.

Table 3. Lamé functions of order $n = 1$ and 2.

$n = 1$

$h_1^{(1)} = 1 + k^2,\ h_2^{(1)} = 1,\ h_{2'}^{(1)} = k^2,$

$f_1^{(1)} = sn(\tau),\ f_2^{(1)} = cn(\tau),\ f_{2'}^{(1)} = dn(\tau).$

$n = 2$

$h_{1,3'}^{(2)} = 2(1 + k^2) \pm 2\sqrt{1 - k^2 + k^4},$

$h_2^{(2)} = 4 + k^2,\ h_{2'}^{(2)} = 1 + 4k^2,\ h_3^{(2)} = 1 + k^2,$

$f_{1,3'}^{(2)} = \frac{1}{3}(1 + k^2 \mp \sqrt{1 - k^2 + k^4}) - k^2 sn^2(\tau),$

$f_2^{(2)} = sn(\tau)cn(\tau),\ f_{2'}^{(2)} = sn(\tau)dn(\tau),\ f_3^{(2)} = cn(\tau)dn(\tau).$

Table 4. Solutions of equation (9) in terms of combinations of Lamé functions listed in table 3.

Interaction type	Combination
$N = 1$	
$(-)$	$(1)_1$
$(+)$	$(2)_1, (2')_1$
$N = 2$	
$(--)$	$(1, 2)_2, (1, 2')_2, (1, 1)_1, (1, 2)_1, (1, 2')_1$
$(-+)$	$(1, 3)_2, (1, 3')_2, (3, 3')_2,$
	$(1, 1)_1, (1, 2)_1, (1, 2')_1, (2, 2)_1, (2, 2')_1, (2', 2')_1$
$(+-)$	$(2, 2')_2, (1, 1)_1, (2, 2)_1, (2, 2')_1, (2', 2')_1$
$(++)$	$(2, 3)_2, (2, 3')_2, (2', 3)_2, (2', 3')_2,$
	$(1, 2)_1, (1, 2')_1, (2, 2)_1, (2, 2')_1, (2', 2')_1$

We list in table 4, for $N = 1$ and 2, the 2^N interaction types, $(\beta_1, \beta_2, \ldots, \beta_N)$, and the corresponding combinations $\left(f_p^{(n)}, f_q^{(n)}, \ldots, f_s^{(n)}\right)$ or $(p, q, \ldots, s)_n$, for $n = 1, \ldots, N$, for the analytic solutions for the N components $(\psi_1, \psi_2, \ldots, \psi_N)$. The amplitudes As needed for equation (14) and the required cs for equation (9), are given in appendix D. They give the complete list of analytic solutions for equation (9), and, using equations (8) and (12), they give the travelling wave solutions ϕ_1 and ϕ_2 for equation (7), for $N = 1$ and 2.

The solutions involving Lamé functions of order 1 for $N = 2$ can be checked to be simply the special cases of the more general solutions given for equation (1) in section 2, by noting equation (3) and identifying

$$\omega_m = K_m + \beta_m \Omega_m^2.$$

It will be noted that for a given interaction type characterized by (β_1, β_2), not every combination of Lamé functions of order $n = 1$ is a possible solution, and this can be checked to be consistent with the conditions that must be satisfied for every solution in section 2, and it is also exhibited by the last column of table 2.

On the other hand, the ten analytic solutions given by Lamé functions of order $n = 2$ for this special L-set of $N = 2$ are not given by the solutions in section 2. Although these solutions for the interaction types $(--)$ and $(++)$ were known previously (some were expressed differently [13–19]), the extension to interaction types $(-+)$ and $(+-)$ 'completes' the analytic solutions for the set the relation of which to the set of CNLS equations that pass the Painlevé test [25] was first pointed out in [21].

The list of solutions given in table 4 and appendix D gives a good overview of the possible analytic solutions for the pairs of coupled waves for the L-set.

We have, for example, from tables 3 and 4, equations (8) and (12), for the interaction type (++) for which both waves propagate in the anomalous GVD regime, a dark–bright wavepair given by the combination of Lamé functions of order 1, $(1, 2)_1$, given by

$$\phi_1 = A_1 sn\{\gamma(t - z/v)\} \exp\left\{ i\left[\frac{t}{2v} - \left(\frac{1}{4v^2} - \omega_1\right) z\right]\right\}$$
$$\phi_2 = A_2 cn\{\gamma(t - z/v)\} \exp\left\{ i\left[\frac{t}{2v} - \left(\frac{1}{4v^2} - \omega_2\right) z\right]\right\}$$

which can be obtained as a special case of solution 5 from section 2.

The same interaction type (++) allows four analytic solutions given by Lamé functions of order 2. For example, we have a second-order wavepair given by $(2, 3)_2$ that consists of

$$\phi_1(z, t) = A_1 sn\{\gamma(t - z/v)\}cn\{\gamma(t - z/v)\} \exp\left\{ i\left[\frac{t}{2v} - \left(\frac{1}{4v^2} - \omega_1\right) z\right]\right\}$$
$$\phi_2(z, t) = A_2 cn\{\gamma(t - z/v)\}dn\{\gamma(t - z/v)\} \exp\left\{ i\left[\frac{t}{2v} - \left(\frac{1}{4v^2} - \omega_2\right) z\right]\right\}.$$

For the 'mixed' interaction type (+−), for which the first wave of the wavepair propagates in the anomalous GVD regime and the second wave in the normal GVD regime, a bright–bright wavepair given by the combination of Lamé functions of order 1, $(2, 2')_1$, is

$$\phi_1 = A_1 cn\{\gamma(t - z/v)\} \exp\left\{ i\left[\frac{t}{2v} - \left(\frac{1}{4v^2} - \omega_1\right) z\right]\right\}$$
$$\phi_2 = A_2 dn\{\gamma(t - z/v)\} \exp\left\{ -i\left[\frac{t}{2v} - \left(\frac{1}{4v^2} + \omega_2\right) z\right]\right\}$$

which is a special case of solution 1 in section 2.

The same interaction type allows one second-order wavepair given by the combination of Lamé functions of order 2, $(2, 2')_2$ that consists of

$$\phi_1(z, t) = A_1 sn\{\gamma(t - z/v)\}cn\{\gamma(t - z/v)\} \exp\left\{ i\left[\frac{t}{2v} - \left(\frac{1}{4v^2} - \omega_1\right) z\right]\right\}$$
$$\phi_2(z, t) = A_2 sn\{\gamma(t - z/v)\}dn\{\gamma(t - z/v)\} \exp\left\{ -i\left[\frac{t}{2v} - \left(\frac{1}{4v^2} + \omega_2\right) z\right]\right\}.$$

The amplitudes As and the parameters ωs for the above waves are all given in appendix D.

5. Coupled-mode equations

The propagation equations governing evolution of the two polarization components along a fibre can take several forms. We consider two of these forms that can be reduced to the forms given in the previous sections.

(1) Consider the following coupled equations:

$$\begin{aligned} &i\Psi_{1Z} \pm \Psi_{1TT} + \eta\Psi_1 + \sigma\Psi_2 + (|\Psi_1|^2 + \nu|\Psi_2|^2)\Psi_1 = 0 \\ &i\Psi_{2Z} \pm \Psi_{2TT} + \eta\Psi_2 + \sigma\Psi_1 + (\nu|\Psi_1|^2 + |\Psi_2|^2)\Psi_2 = 0. \end{aligned} \tag{15}$$

It is known [13, 14] that by substituting

$$\Psi_{1,2} = \phi_1 \pm i\phi_2 \tag{16}$$

and

$$z = (\nu + 1)Z \qquad t = \sqrt{\nu + 1}T \tag{17}$$

equation (15) can be transformed into

$$\mathrm{i}\phi_{mz} \pm \phi_{mtt} + \kappa_m\phi_1 + p(|\phi_1|^2 + |\phi_2|^2)\phi_m + q\left(\phi_1^2 + \phi_2^2\right)\phi_m^* = 0 \qquad m = 1, 2 \tag{18}$$

where $p = 2/(\nu + 1)$, $q = (\nu - 1)/(\nu + 1)$, (note that $p + q = 1$), $\kappa_1 = (\eta + \sigma)/(\nu + 1)$, $\kappa_2 = (\eta - \sigma)/(\nu + 1)$. The case $\eta = 0$, $p = \frac{2}{3}$, $q = \frac{1}{3}$, $\nu = 2$, $\kappa_1 = -\kappa_2 = \sigma/3$, is applicable to birefringent fibre [1–3].

(2) Consider the following coupled equations,

$$\mathrm{i}\Phi_{mz} \pm \Phi_{mtt} + p(|\Phi_1|^2 + |\Phi_2|^2)\Phi_m + q\left(\Phi_1^2\,\mathrm{e}^{2\mathrm{i}\kappa_1 z} + \Phi_2^2\,\mathrm{e}^{2\mathrm{i}\kappa_2 z}\right)\Phi_m^*\,\mathrm{e}^{-2\mathrm{i}\kappa_m z} = 0 \qquad m = 1, 2 \tag{19}$$

where $p + q = 1$. It is also known [4, 5] that they can also be transformed into equation (18) by the substitutions

$$\Phi_m = \phi_m\,\mathrm{e}^{-\mathrm{i}\kappa_m z} \qquad m = 1, 2. \tag{20}$$

Thus for both equations (15) and (19), we may consider equation (18) as a starting point. By making the substitution $\phi_m(z, t) = \psi_m(t)\,\mathrm{e}^{\mathrm{i}\omega_m z}$, where ω_m is a real constant and $\psi_m(t)$ are real functions of t only, the following coupled equations for $\psi_m(t)$ are obtained,

$$\pm\psi_{mtt} + c_m\psi_m + \left(\psi_1^2 + \psi_2^2\right)\psi_m = 0 \qquad m = 1, 2 \tag{21}$$

where $c_m = \kappa_m - \omega_m$. Equation (21) is a special case of equation (9) for which the analytic solutions are given in appendix D.

6. Summary

We have presented the following results:

(1) A set of six analytic solutions for a general set of two CNKG equations, equation (1). They are classified into three groups of periodic wavepairs: one kind of bright–bright, three kinds of grey–grey and two kinds of bright–grey wavepairs. All six analytic solutions are new even though some special cases of them are well known. The possibilities of these wavepairs propagating in various combinations of anomalous and normal GVD regimes for the case of two CNLS equations are presented with explicit conditions that are applicable not only for the physically applicable regime $\nu > 0$ but also for the (presently) unphysical regime $\nu < 0$. Consideration of the entire range of ν gives us a better perspective about the symmetry regarding the possible propagation of wavepairs that are of groups I and II (bright–bright and grey–grey) and of those that are of groups III (bright–grey).

(2) A new superposition solution for a special case of equation (1) which we called the symmetric case.

(3) Ten additional analytic solutions in terms of Lamé functions of order 2 for another special case of two CNLS equations which we called the L-set. While only four of these solutions that have 'mixed' interactions are new, their inclusion completes the analytic solutions for the set that passes the Painlevé test.

The stability of some special cases of our periodic wavepairs that are non-periodic has been studied [12], but a systematic stability analysis of the more general periodic solitary wavepairs has not been done. The richness and 'completeness' of these periodic solitary wavepairs should invite more theoretical study of their stability property, as well as their experimental production and testing similar to that [31] for the periodic waves that are solutions of Bloch–Maxwell equations.

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Appendix A

In this appendix, we first consider a one-component nonlinear Klein–Gordon equation given by

$$\mathrm{i}\phi_z + \alpha''\phi_{zz} + \mathrm{i}\beta'\phi_t + \beta''\phi_{tt} + \kappa\phi + R|\phi|^2\phi = 0 \tag{A.1}$$

and describe its analytic solitary-wave solutions, as it is instructive to see, by comparison, their difference with the coupled solitary-wave solutions of the two-component CNKG equations (1).

We let $\phi(z,t) = \rho(z,t)\,\mathrm{e}^{\mathrm{i}\theta(z,t)}$, $j = 1, 2$, and substitute this into equation (A.1). Separating out the real and imaginary parts, we get two differential equations. By assuming that $\rho = \rho(\xi)$ and $\theta = Kz - \Omega t + \chi(\xi)$, where $\xi \equiv t - z/v$, and making a transformation from z, t to ξ such that $\partial/\partial t = \mathrm{d}/\mathrm{d}\xi$, $\partial/\partial z = -v^{-1}\mathrm{d}/\mathrm{d}\xi$, then choosing v such that

$$v^{-1} = -2\alpha'' K v^{-1} + \beta' - 2\beta''\Omega \tag{A.2}$$

implies that $\mathrm{d}\chi/\mathrm{d}\xi = (\mathrm{const})/\rho^2(\xi)$. This choice of v also results in an ordinary second-order differential equations (with respect to ξ) for $\rho(\xi)$. Now letting $\rho(\xi) = Af(\xi) = A[1 - C^2 g^2(\xi)]^{1/2}$, where

$$g = sn(\gamma\xi),\ \ cn(\gamma\xi) \text{ or } dn(\gamma\xi) \tag{A.3}$$

results in a differential equation for f in the form

$$\mathrm{d}^2 f/\mathrm{d}\xi^2 + bf + cf^3 - df^{-3} = 0. \tag{A.4}$$

Substituting $f(\xi) = [1 - C^2 g^2(\xi)]^{1/2}$ into the above equation and solving a set of algebraic equations give two equations that relate various parameters, and the constant appearing in $\mathrm{d}\chi/\mathrm{d}\xi$ is identified with $d^{1/2}A^2$, i.e. we have

$$\mathrm{d}\chi/\mathrm{d}\xi = \mathrm{d}^{1/2}A^2/\rho^2(\xi).$$

The equations or conditions that accompany each solution can be expressed more concisely if we make the following notations.

Denote

$$a \equiv \alpha'' v^{-2} + \beta''$$
$$b \equiv a^{-1}(\kappa - K + \beta'\Omega - \alpha'' K^2 - \beta''\Omega^2)$$
$$c \equiv a^{-1} R A^2.$$

The three possible solutions of equation (A.1) are now given by

$$\phi(z,t) = A[1 - C^2 g^2(\xi)]^{1/2} \exp(\mathrm{i}[Kz - \Omega t + \chi(\xi)]) \tag{A.5}$$

where $g(\xi)$ is given by equation (A.3), and

$$\chi(\xi) = d^{1/2} \int^{\xi} [1 - C^2 g^2(\xi)]^{-1} \,\mathrm{d}\xi. \tag{A.6}$$

Equation (A.2) is the first condition that accompanies each of these solutions. The solutions, the second and third conditions that accompany each solution, and the d for $\chi(\xi)$, can be expressed concisely as follows:

Solution 1. $g = sn(\gamma\xi)$ for equation (A.5),

$$b = [1 - (3C^{-2} - 1)k^2]\gamma^2$$
$$c = 2C^{-2}k^2\gamma^2$$
$$d = (1 - C^2)(1 - C^{-2}k^2) \qquad k^2 \leqslant C^2 \leqslant 1.$$

Solution 2. $g = cn(\gamma\xi)$ for equation (A.5),

$$b = [1 + (3C^{-2} - 2)k^2]\gamma^2$$
$$c = -2C^{-2}k^2\gamma^2$$
$$d = (1 - C^2)[1 + C^{-2}k^2(1 - C^2)] \qquad 0 < C^2 \leqslant 1.$$

Solution 3. $g = dn(\gamma\xi)$ for equation (A.5),

$$b = [k^2 + (3C^{-2} - 2)]\gamma^2$$
$$c = -2C^{-2}\gamma^2$$
$$d = (1 - C^2)[k^2 + C^{-2}(1 - C^2)] \qquad 0 < C^2 \leqslant 1.$$

The three solutions can be divided into two groups: group I consists of one kind of bright periodic solitary wave (solution 1), and group II consists of two kinds of grey periodic solitary wave (solutions 2 and 3). The condition given by the equation involving c can be used to immediately identify the possible solutions for a given set of parameters given in equation (A.1) by requiring that A^2 must be > 0. Thus, for example, for $\alpha'' = 0$, and $R > 0$, equation (A.1) with $\beta'' > 0$ (the anomalous GVD regime) allows solution 1 (the bright solitary wave) but not solution 2 or 3 (the grey solitary wave); while equation (A.1) with $\beta'' < 0$ (the normal GVD regime) allows solutions 2 and 3 (the grey solitary waves) but not solution 1. On the other hand, it will be seen in the following that the two-component CNLS equations allow more flexibilities and unexpected results.

Let us now consider the two-component CNKG equations (1) and outline the steps leading to the analytic solutions of equation (1) which we present in section 2.

We assume $\phi_j(z,t) = \rho_j(z,t)\,\mathrm{e}^{\mathrm{i}\theta_j(z,t)}$, $j = 1, 2$, and substitute them into equation (1). Separating out the real and imaginary parts, we get four differential equations. We assume that $\rho_j = \rho_j(\xi)$ and $\theta_j = K_j z - \Omega_j t + \chi_j(\xi)$, and make a transformation from z, t to ξ as before. Then the choice of v such that

$$v^{-1} = -2\alpha_1'' K_1 v^{-1} + \beta_1' - 2\beta_1''\Omega_1 = -2\alpha_2'' K_2 v^{-1} + \beta_2' - 2\beta_2''\Omega_2 \tag{A.7}$$

implies that $\mathrm{d}\chi_j/\mathrm{d}\xi = (\mathrm{const})/\rho_j^2(\xi)$. The two equations for v are the first two equations that accompany each solution in section 2. This choice of v also results in two coupled ordinary second-order differential equations for $\rho_j(\xi)$. Now letting $\rho_j(\xi) = A_j f_j(\xi) = A\left[1 - C_j^2 g_j^2(\xi)\right]^{1/2}$, where g_j are given in table 1, result in two uncoupled differential equations for f_j in the form

$$\mathrm{d}^2 f_j/\mathrm{d}\xi^2 + b_j f_j + c_j f_j^3 - d_j f_j^{-3} = 0 \qquad j = 1, 2. \tag{A.8}$$

Substituting the g_j or f_j given in table 1 into equation (A.8) and solving a set of algebraic equations give the four remaining conditions that accompany each solution: the left-hand sides of the third, fourth, fifth and sixth equations are the expressions for b_1, b_2, c_1 and c_2 respectively, in the above equation, and the constant appearing in $\mathrm{d}\chi_j/\mathrm{d}\xi$ is identified with $d_j^{1/2}A_j^2$, i.e. we have

$$\mathrm{d}\chi_j/\mathrm{d}\xi = d_j^{1/2}A_j^2\big/\rho_j^2(\xi).$$

These equations that accompany each solution can be expressed more concisely if we make the following notations.

(1) For $g_1 = g_2$.

With a_j given by equation (4), denote

$$\begin{aligned}
b_1 &\equiv a_1^{-1}\big[\kappa_1 - K_1 + \beta_1'\Omega_1 - \alpha_1''K_1^2 - \beta_1''\Omega_1^2 + \nu R_1 A_2^2\big(1 - C_2^2\big/C_1^2\big)\big]\\
b_2 &\equiv a_2^{-1}\big[\kappa_2 - K_2 + \beta_2'\Omega_2 - \alpha_2''K_2^2 - \beta_2''\Omega_2^2 + \nu R_2 A_1^2\big(1 - C_1^2\big/C_2^2\big)\big]\\
c_1 &\equiv a_1^{-1}R_1\big(A_1^2 + \nu A_2^2 C_2^2\big/C_1^2\big)\\
c_2 &\equiv a_2^{-1}R_2\big(A_2^2 + \nu A_1^2 C_1^2\big/C_2^2\big).
\end{aligned}$$

(2) For $g_1 \neq g_2$.

We express the relationship between$\rho_1(\xi)$and $\rho_2(\xi)$ in the form $\rho_2^2 + \epsilon\rho_1^2 = E$.

Denote

$$\begin{aligned}
b_1 &\equiv a_1^{-1}\big(\kappa_1 - K_1 + \beta_1'\Omega_1 - \alpha_1''K_1^2 - \beta_1''\Omega_1^2 + \nu R_1 E\big)\\
b_2 &\equiv a_2^{-1}\big(\kappa_2 - K_2 + \beta_2'\Omega_2 - \alpha_2''K_2^2 - \beta_2''\Omega_2^2 + \nu R_2 \epsilon^{-1} E\big)\\
c_1 &\equiv a_1^{-1}R_1 A_1^2(1 - \nu\epsilon)\\
c_2 &\equiv a_2^{-1}R_2 A_2^2(1 - \nu\epsilon^{-1}).
\end{aligned}$$

(i) For $g_1(\xi) = cn(\gamma\xi)$, $g_2(\xi) = dn(\gamma\xi)$

$$\epsilon = -k^2C_2^2C_1^{-2}A_2^2A_1^{-2},\ E = A_2^2\big[1 - C_2^2 + k^2\big(C_2^2 - C_2^2C_1^{-2}\big)\big].$$

(ii) For $g_1(\xi) = sn(\gamma\xi)$, $g_2(\xi) = cn(\gamma\xi)$

$$\epsilon = C_2^2C_1^{-2}A_2^2A_1^{-2},\ E = A_2^2\big(1 - C_2^2 + C_2^2C_1^{-2}\big).$$

(iii) For $g_1(\xi) = sn(\gamma\xi)$, $g_2(\xi) = dn(\gamma\xi)$

$$\epsilon = k^2C_2^2C_1^{-2}A_2^2A_1^{-2},\ E = A_2^2\big(1 - C_2^2 + k^2C_2^2C_1^{-2}\big).$$

The six possible solutions of equation (1) are now given by equation (3), and the conditions that accompany each solution in section 2 represented by the last four equations and the expression for χ_j can be expressed concisely as follows:

(1) For $g_j = sn(\gamma\xi)$

$$\begin{aligned}
b_j &= \big[1 - \big(3C_j^{-2} - 1\big)k^2\big]\gamma^2\\
c_j &= 2C_j^{-2}k^2\gamma^2\\
d_j &= \big(1 - C_j^2\big)\big(1 - C_j^{-2}k^2\big) \qquad k^2 \leqslant C_j^2 \leqslant 1.
\end{aligned}$$

(2) For $g_j = cn(\gamma\xi)$

$$\begin{aligned}
b_j &= \big[1 + \big(3C_j^{-2} - 2\big)k^2\big]\gamma^2\\
c_j &= -2C_j^{-2}k^2\gamma^2\\
d_j &= \big(1 - C_j^2\big)\big[1 + C_j^{-2}k^2\big(1 - C_j^2\big)\big] \qquad 0 < C_j^2 \leqslant 1.
\end{aligned}$$

(3) For $g_j = dn(\gamma\xi)$

$$b_j = \left[k^2 + \left(3C_j^{-2} - 2\right)\right]\gamma^2$$
$$c_j = -2C_j^{-2}\gamma^2$$
$$d_j = \left(1 - C_j^2\right)\left[k^2 + C_j^{-2}\left(1 - C_j^2\right)\right] \qquad 0 < C_j^2 \leqslant 1.$$

From the expressions for c_1 and c_2, and from the conditions given by the equations involving c_1 and c_2, many more possibilities than the one-component case are seen to be realizable, some of which are discussed in section 2. The mathematical origin that gives rise to the many more possibilities compared to the one-component case is clearly shown from above: the coupling between the two waves could result in an anomalous GVD regime $\beta_1'' > 0$ permitting a grey solitary wave with $g_1 = cn(\gamma\xi)$ or $dn(\gamma\xi)$, and a normal GVD regime $\beta_2'' < 0$ permitting a bright solitary wave with $g_2 = sn(\gamma\xi)$.

Appendix B

In this appendix, we present, assuming that $\alpha_1'' = \alpha_2'' = 0$, R_1 and R_2 both > 0 in equation (1), the list of possible propagations of bright–bright (solution 1), grey–grey (solutions 2, 3 and 4) and bright-grey (solutions 5 and 6) in various GVD regimes for the two individual waves of the wavepair.

(1) Bright–bright periodic wavepair (solution 1):

(i) $\beta_1'' > 0$, $\beta_2'' > 0$: possible for $\nu > -1$:
Always possible for $-1 < \nu \leqslant 0$.
Possible for $0 \leqslant \nu < 1$, if $|\nu|^{-1}|\beta_2''|R_2^{-1} > |\beta_1''|R_1^{-1} > |\nu||\beta_2''|R_2^{-1}$.
Possible for $\nu = 1$, if $|\beta_1''|R_1^{-1} = |\beta_2''|R_2^{-1}$.
Possible for $\nu > 1$, if $|\nu||\beta_2''|R_2^{-1} > |\beta_1''|R_1^{-1} > |\nu|^{-1}|\beta_2''|R_2^{-1}$.

(ii) $\beta_1'' > 0$, $\beta_2'' < 0$: Possible for $\nu < 0$:
Possible for $-1 < \nu < 0$, if $|\beta_1''|R_1^{-1} > |\nu|^{-1}|\beta_2''|R_2^{-1}$.
Possible for $\nu = -1$, if $|\beta_1''|R_1^{-1} = |\beta_2''|R_2^{-1}$.
Possible for $\nu < -1$, $|\beta_1''|R_1^{-1} < |\nu|^{-1}|\beta_2''|R_2^{-1}$.

(iii) $\beta_1'' < 0$, $\beta_2'' > 0$: possible for $\nu < 0$:
Possible for $-1 < \nu < 0$, if $|\beta_1''|R_1^{-1} < |\nu||\beta_2''|R_2^{-1}$.
Possible for $\nu = -1$, if $|\beta_1''|R_1^{-1} = |\beta_2''|R_2^{-1}$.
Possible for $\nu < -1$, if $|\beta_1''|R_1^{-1} > |\nu||\beta_2''|R_2^{-1}$.

(iv) $\beta_1'' < 0$, $\beta_2'' < 0$: always possible for $\nu < -1$.

(2) Grey–grey periodic wavepair (solutions 2, 3 and 4):

(i) $\beta_1'' > 0$, $\beta_2'' > 0$: always possible for $\nu < -1$.

(ii) $\beta_1'' > 0$, $\beta_2'' < 0$: possible for $\nu < 0$:
Possible for $-1 < \nu < 0$, if $|\beta_1''|R_1^{-1} < |\nu||\beta_2''|R_2^{-1}$.
Possible for $\nu = -1$, if $|\beta_1''|R_1^{-1} = |\beta_2''|R_2^{-1}$.
Possible for $\nu < -1$, if $|\beta_1''|R_1^{-1} > |\nu||\beta_2''|R_2^{-1}$.

(iii) $\beta_1'' < 0, \beta_2'' > 0$: possible for $\nu < 0$:

Possible for $-1 < \nu < 0$, if $|\beta_1''|R_1^{-1} > |\nu|^{-1}|\beta_2''|R_2^{-1}$.

Possible for $\nu = -1$, if $|\beta_1''|R_1^{-1} = |\beta_2''|R_2^{-1}$.

Possible for $\nu < -1$, if $|\beta_1''|R_1^{-1} < |\nu|^{-1}|\beta_2''|R_2^{-1}$.

(iv) $\beta_1'' < 0, \beta_2'' < 0$: possible for $\nu > -1$:

Always possible for $-1 < \nu \leqslant 0$.

Possible for $0 \leqslant \nu < 1$, if $|\nu|^{-1}|\beta_2''|R_2^{-1} > |\beta_1''|R_1^{-1} > |\nu||\beta_2''|R_2^{-1}$.

Possible for $\nu = 1$, if $|\beta_1''|R_1^{-1} = |\beta_2''|R_2^{-1}$.

Possible for $\nu > 1$, if $|\nu|^{-1}|\beta_2''|R_2^{-1} < |\beta_1''|R_1^{-1} < |\nu||\beta_2''|R_2^{-1}$.

(3) Bright–grey periodic wavepair (solutions 5 and 6)

(i) $\beta_1'' > 0, \beta_2'' > 0$: possible for $\nu > 0$:

Possible for $0 < \nu < 1$, if $|\beta_1''|R_1^{-1} > |\nu|^{-1}|\beta_2''|R_2^{-1}$.

Possible for $\nu = 1$, if $|\beta_1''|R_1^{-1} = |\beta_2''|R_2^{-1}$.

Possible for $\nu > 1$, if $|\beta_1''|R_1^{-1} < |\nu|^{-1}|\beta_2''|R_2^{-1}$.

(ii) $\beta_1'' > 0, \beta_2'' < 0$: possible for $\nu < 1$:

Always possible for $0 \leqslant \nu < 1$.

Possible for $-1 < \nu \leqslant 0$, if $|\nu|^{-1}|\beta_2''|R_2^{-1} > |\beta_1''|R_1^{-1} > |\nu||\beta_2''|R_2^{-1}$.

Possible for $\nu = -1$, if $|\beta_1''|R_1^{-1} = |\beta_2''|R_2^{-1}$.

Possible for $\nu < -1$, if $|\nu||\beta_2''|R_2^{-1} > |\beta_1''|R_1^{-1} > |\nu|^{-1}|\beta_2''|R_2^{-1}$.

(iii) $\beta_1'' < 0, \beta_2'' > 0$: always possible for $\nu > 1$.

(iv) $\beta_1'' < 0, \beta_2'' < 0$: possible for $\nu > 0$:

Possible for $0 < \nu < 1$, if $|\beta_1''|R_1^{-1} < |\nu||\beta_2''|R_2^{-1}$.

Possible for $\nu = 1$, if $|\beta_1''|R_1^{-1} = |\beta_2''|R_2^{-1}$.

Possible for $\nu > 1$, if $|\beta_1''|R_1^{-1} > |\nu||\beta_2''|R_2^{-1}$.

Appendix C

Galilean invariance for a standard nonlinear Schrödinger equation is well known [1–3, 6]. The same principle can be easily extended to a more general form of N coupled nonlinear Schrödinger equations with arbitrary coefficients, but the precise form of this more general form of transformation is not easily found in the published texts. We state below this general form for present and future reference, a specific form of which was made use of in section 4.

If $\phi_m(z, t)$ is a solution of the equation

$$\mathrm{i}\partial\phi_m/\partial z + \mathrm{i}\beta_m'\partial\phi_m/\partial t + \beta_m''\partial^2\phi_m/\partial t^2 + F(|\phi_1|^2, \ldots, |\phi_N|^2)\phi_m = 0 \qquad m = 1, \ldots, N$$

where $F(|\phi_1|^2, \ldots, |\phi_N|^2)$ is an arbitrary function of $|\phi_1|^2, \ldots, |\phi_N|^2$, then $\phi_m(z, t - z/v) \exp\left\{\mathrm{i}\frac{1}{2\beta_m''v}\left[t - \left(\frac{1}{2v} + \beta_m'\right)z\right]\right\}$ is also a solution of the equation.

The Galilean transformation is useful for 'boosting' a stationary-wave solution of a general set of N coupled nonlinear Schrödinger equations into a travelling-wave solution.

Appendix D

In this appendix, we list the required As and cs for the analytic solutions of equation (9) in terms of equation (14) and Lamé functions of order n for $N = 1$ and 2, $n \leqslant N$, given in table 3. γ is a scaling parameter.

$N = 1$

$(-)$ $(1)_1$ $A_1^2 = 2k^2\gamma^2$ $c_1 = (1 + k^2)\gamma^2$

$(+)$ $(2)_1$ $A_1^2 = 2k^2\gamma^2$ $c_1 = (1 - 2k^2)\gamma^2$

$(2')_1$ $A_1^2 = 2\gamma^2$ $c_1 = -(2 - k^2)\gamma^2.$

$N = 2$

$(--)$

$(1, 2)_2$

$$A_1^2 = \frac{18\gamma^2}{-(2-k^2) + 2\sqrt{1-k^2+k^4}} \qquad A_2^2 = \frac{18k^4\gamma^2}{-(2-k^2) + 2\sqrt{1-k^2+k^4}}$$

$$c_1 = \frac{2\gamma^2}{-(2-k^2) + 2\sqrt{1-k^2+k^4}}\left\{2 - 2k^2 + 5k^4 - (2-k^2)\sqrt{1-k^2+k^4}\right\}$$

$$c_2 = \frac{\gamma^2}{-(2-k^2) + 2\sqrt{1-k^2+k^4}}\left\{-4 + 4k^2 + 5k^4 + 2(2-k^2)\sqrt{1-k^2+k^4}\right\}.$$

$(1, 2')_2$

$$A_1^2 = \frac{18\gamma^2}{1 - 2k^2 + 2\sqrt{1-k^2+k^4}} \qquad A_2^2 = \frac{18k^2\gamma^2}{1 - 2k^2 + 2\sqrt{1-k^2+k^4}}$$

$$c_1 = \frac{2\gamma^2}{1 - 2k^2 + 2\sqrt{1-k^2+k^4}}\left\{5 - 2k^2 + 2k^4 + (1-2k^2)\sqrt{1-k^2+k^4}\right\}$$

$$c_2 = \frac{\gamma^2}{1 - 2k^2 + 2\sqrt{1-k^2+k^4}}\left\{5 + 4k^2 - 4k^4 - 2(1-2k^2)\sqrt{1-k^2+k^4}\right\}.$$

$(1, 1)_1$

$$A_1^2 + A_2^2 = 2k^2\gamma^2 \qquad c_1 = c_2 = (1 + k^2)\gamma^2.$$

$(1, 2)_1$

$$A_1^2 = c_1 - (1 - k^2)\gamma^2 \qquad A_2^2 = c_1 - (1 + k^2)\gamma^2 \qquad c_1 - c_2 = k^2\gamma^2.$$

$(1, 2')_1$

$$A_1^2 = k^2\{c_1 + (1 - k^2)\gamma^2\} \qquad A_2^2 = c_1 - (1 + k^2)\gamma^2 \qquad c_1 - c_2 = \gamma^2.$$

$(-+)$

$(1, 3)_2$

$$A_1^2 = \frac{18\gamma^2}{1 + k^2 + 2\sqrt{1-k^2+k^4}} \qquad A_2^2 = \frac{18k^2\gamma^2}{1 + k^2 + 2\sqrt{1-k^2+k^4}}$$

$$c_1 = \frac{2\gamma^2}{1 + k^2 + 2\sqrt{1-k^2+k^4}}\left\{5 - 8k^2 + 5k^4 + (1+k^2)\sqrt{1-k^2+k^4}\right\}$$

$$c_2 = \frac{\gamma^2}{1 + k^2 + 2\sqrt{1-k^2+k^4}}\left\{5 - 14k^2 + 5k^4 - 2(1+k^2)\sqrt{1-k^2+k^4}\right\}.$$

$(1, 3')_2$

$$A_1^2 = A_2^2 = \frac{9\gamma^2}{2\sqrt{1-k^2+k^4}}$$

$$c_1 = -c_2 = 2\gamma^2\sqrt{1-k^2+k^4}.$$

$(3, 3')_2$

$$A_1^2 = \frac{18k^2\gamma^2}{-(1+k^2)+2\sqrt{1-k^2+k^4}} \qquad A_2^2 = \frac{18\gamma^2}{-(1+k^2)+2\sqrt{1-k^2+k^4}}$$

$$c_1 = \frac{\gamma^2}{1+k^2-2\sqrt{1-k^2+k^4}}\left\{5-14k^2+5k^4+2(1+k^2)\sqrt{1-k^2+k^4}\right\}$$

$$c_2 = \frac{2\gamma^2}{1+k^2-2\sqrt{1-k^2+k^4}}\left\{5-8k^2+5k^4-(1+k^2)\sqrt{1-k^2+k^4}\right\}.$$

$(1, 1)_1$

$$A_1^2 - A_2^2 = 2k^2\gamma^2 \qquad c_1 = c_2 = (1+k^2)\gamma^2.$$

$(1, 2)_1$

$$A_1^2 = c_1 - (1-k^2)\gamma^2 \qquad A_2^2 = -c_1 + (1+k^2)\gamma^2 \qquad c_1 - c_2 = k^2\gamma^2.$$

$(1, 2')_1$

$$A_1^2 = k^2\{c_1 - (1-k^2)\gamma^2\} \qquad A_2^2 = -c_1 + (1+k^2)\gamma^2 \qquad c_1 - c_2 = \gamma^2.$$

$(2, 2)_1$

$$-A_1^2 + A_2^2 = 2k^2\gamma^2 \qquad c_1 = c_2 = (1-2k^2)\gamma^2.$$

$(2, 2')_1$

$$A_1^2 = k^2k'^{-2}\{-c_1 - \gamma^2\} \qquad A_2^2 = k'^{-2}\{-c_1 + (1-2k^2)\gamma^2\} \qquad c_1 - c_2 = k'^2\gamma^2.$$

$(2', 2')_1$

$$-A_1^2 + A_2^2 = 2\gamma^2 \qquad c_1 = c_2 = -(2-k^2)\gamma^2.$$

$(+-)$

$(2, 2')_2$

$$A_1^2 = 6k^4k'^{-2}\gamma^2 \qquad A_2^2 = 6k^2k'^{-2}\gamma^2$$

$$c_1 = (4+k^2)\gamma^2$$

$$c_2 = (1+4k^2)\gamma^2.$$

$(1, 1)_1$

$$-A_1^2 + A_2^2 = 2k^2\gamma^2 \qquad c_1 = c_2 = (1+k^2)\gamma^2.$$

$(2, 2)_1$

$$A_1^2 - A_2^2 = 2k^2\gamma^2 \qquad c_1 = c_2 = (1-2k^2)\gamma^2.$$

$(2, 2')_1$

$$A_1^2 = k^2k'^{-2}\{c_1 + \gamma^2\} \qquad A_2^2 = k'^{-2}\{c_1 - (1-2k^2)\gamma^2\}. \qquad c_1 - c_2 = k'^2\gamma^2.$$

$(2', 2')_1$

$$A_1^2 - A_2^2 = 2\gamma^2 \qquad c_1 = c_2 = -(2-k^2)\gamma^2.$$

(++)

$(2,3)_2$

$A_1^2 = 6k^4\gamma^2 \qquad A_2^2 = 6k^2\gamma^2$

$c_1 = (4-5k^2)\gamma^2$

$c_2 = (1-5k^2)\gamma^2.$

$(2,3')_2$

$$A_1^2 = \frac{18k^4\gamma^2}{2-k^2+2\sqrt{1-k^2+k^4}} \qquad A_2^2 = \frac{18\gamma^2}{2-k^2+2\sqrt{1-k^2+k^4}}$$

$$c_1 = \frac{-\gamma^2}{2-k^2+2\sqrt{1-k^2+k^4}}\left\{-4+4k^2+5k^4-2(2-k^2)\sqrt{1-k^2+k^4}\right\}$$

$$c_2 = \frac{-2\gamma^2}{2-k^2+2\sqrt{1-k^2+k^4}}\left\{2-2k^2+5k^4+(2-k^2)\sqrt{1-k^2+k^4}\right\}.$$

$(2',3)_2$

$A_1^2 = A_2^2 = 6\gamma^2$

$c_1 = -(5-4k^2)\gamma^2$

$c_2 = -(5-k^2)\gamma^2.$

$(2',3')_2$

$$A_1^2 = \frac{18k^2\gamma^2}{-1+2k^2+2\sqrt{1-k^2+k^4}} \qquad A_2^2 = \frac{18\gamma^2}{-1+2k^2+2\sqrt{1-k^2+k^4}}$$

$$c_1 = \frac{-\gamma^2}{-1+2k^2+2\sqrt{1-k^2+k^4}}\left\{5+4k^2-4k^4-2(-1+2k^2)\sqrt{1-k^2+k^4}\right\}$$

$$c_2 = \frac{-2\gamma^2}{-1+2k^2+2\sqrt{1-k^2+k^4}}\left\{5-2k^2+2k^4+(-1+2k^2)\sqrt{1-k^2+k^4}\right\}.$$

$(1,2)_1$

$A_1^2 = -c_1 + (1-k^2)\gamma^2 \qquad A_2^2 = -c_1 + (1+k^2)\gamma^2 \qquad c_1 - c_2 = k^2\gamma^2.$

$(1,2')_1$

$A_1^2 = k^2\{-c_1 - (1-k^2)\gamma^2\} \qquad A_2^2 = -c_1 + (1+k^2)\gamma^2 \qquad c_1 - c_2 = \gamma^2.$

$(2,2)_1$

$A_1^2 + A_2^2 = 2k^2\gamma^2 \qquad c_1 = c_2 = (1-2k^2)\gamma^2.$

$(2,2')_1$

$A_1^2 = k^2k'^{-2}\{c_1+\gamma^2\} \qquad A_2^2 = k'^{-2}\{-c_1+(1-2k^2)\gamma^2\} \qquad c_1 - c_2 = k'^2\gamma^2.$

$(2',2')_1$

$A_1^2 + A_2^2 = 2\gamma^2 \qquad c_1 = c_2 = -(2-k^2)\gamma^2.$

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